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Optimization Models

one variable optimization and multivariable optimization

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Mathematical Optimization

- Problems in optimization are the most important applications of mathematics
- Example in computer networks;
  - Communication: maximize throughput and minimize delays
  - Routing: find the shortest path
  - Multi-processor multi-core system: minimize the waiting time of tasks
  - P2P system: minimize searching cost
- Something in common:
  - A particular mathematical structure
  - Control variables (to be determined)
  - Constraints
  - Optimization targets (measurable)

The Five Step Method

- 1. Ask the question
- 2. Select the modeling approach
- 3. Formulate the model
- 4. Solve the model
- 5. Answer the question

One Variable Optimization

Example - Pig Selling Problem

- Pig selling problem
  - A pig weight 200 pounds, gains 5 pounds per day, and costs 45 cents a day to keep.
  - The market price is 65 cent per pound, but falling 1 cent per day.
  - When should the pig be sold?

- Step 1: Ask the question
- Variables:

  t = time (days)

  w = weight (lbs)

  p = price ($/lb)

  c = cost of keeping pig t days ($)

  Pr = profit from sale of pig ($)
- Assumptions and constraints:

  $w(t)=200+5t$

  $p(t)=0.65-0.01t$

  $c(t)=0.45t$

  $Pr= w(t)*p(t)-c(t)$

  $t\geq 0$
- Objective:

  Maximize Pr

- Step 2: Select the modeling approach
- Basic calculus
  - Given a function $y=f(x)$, if f attains its extreme value at x, then $f'(x)=0$

- Formulate the model
- According to step 1

$$Pr = w(t) * p(t) - c(t) = (200 + 5t)(0.65 - 0.01t) - 0.45t$$

- Let $y = Pr, x = t,$
- The problem is to maximize

$$y = f(x) = (200 + 5x)(0.65 - 0.01x) - 0.45x$$

over the set $S = \{x : x \geq 0\}$

- Step 4: Solve the model

f(x) if differentiable:

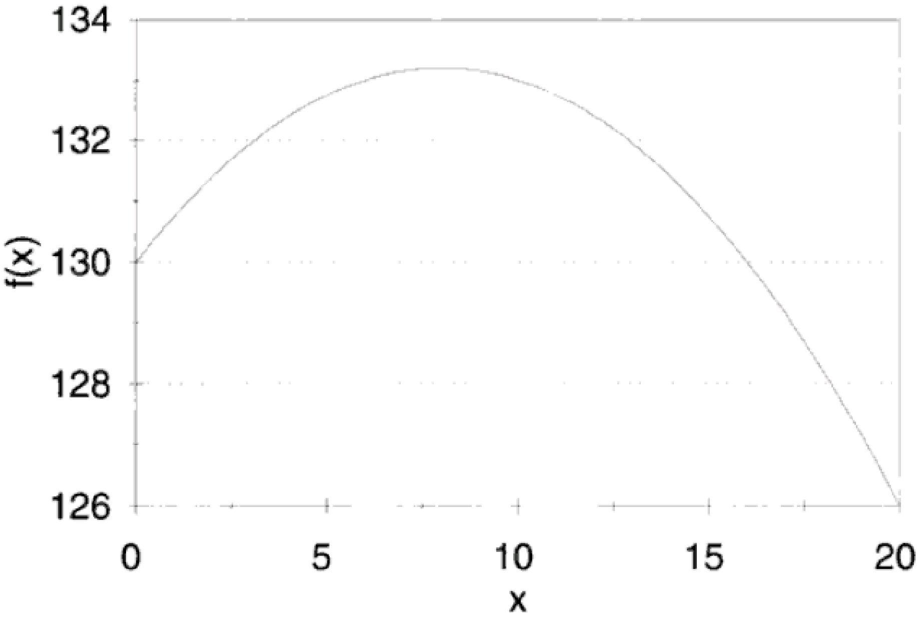
$$f'(x) = \frac{8 - x}{10}$$

Let $f'(x) = 0$

we have x=8

- The global maximum is

x=8,y=f(8)=133.20



- Step 5: Answer the question
- Question: when to sell the pig?
- Answer:

  According to our mathematical model, we should sell it after 8 days, which obtains a net profit of $133.20.

Summary of 5 step method

- 1. Ask the question
  - Make a list of variables
  - State all assumptions and constraints about the variables, including equations and inequations
  - State the objective of the problem in math terms
- 2. Select the modeling approach
  - Choose a general solution procedure for solving the problem
  - This require experience, skill, and mathematical knowledge.
- 3. Formulate the model
  - Restate the questions in step 1 to match the model in step 2
- 4. Solve the model
  - Apply the general solution procedure to the specific problem
- 5. Answer the question
  - Rephrase the results of step 4 in nontechnical terms (to make it easy to understand)

Sensitive Analysis

- For the real world situation, some factors are uncertained
- How sensitive our conclusions are to each of the assumptions we have made?
- For example,
  - What is the impact of the growth rate of pig?
  - What is the impact of price falling rate?

$$Pr = w(t) * p(t) - c(t) = (200 + 5t)(0.65 - 0.01t) - 0.45t$$

- (1) Impact of price falling rate r:

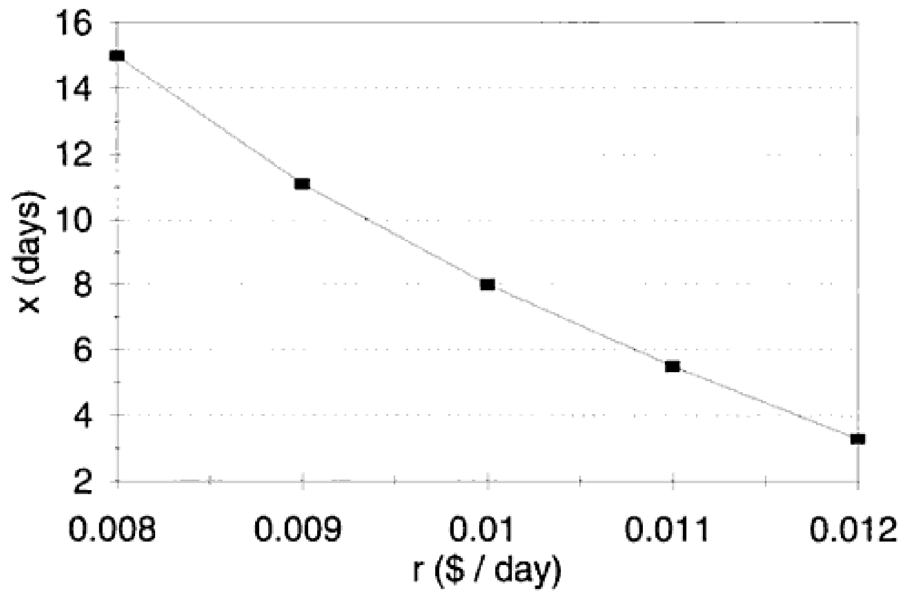
$$p = 0.65 - rt$$

$$y = f(x) = (0.65 - rx)(200 + 5x) - 0.45x$$

$$f'(x) = \frac{-2(25rx + 500r - 7)}{5} = 0$$

$$x = \frac{7 - 500r}{25r}$$

- As long as 0<r<0.014, f'>0, the optimal time is given by x, otherwise, f'<0, the profit decreasing on [0,∞)



- (2)Impact of the growth rate g.

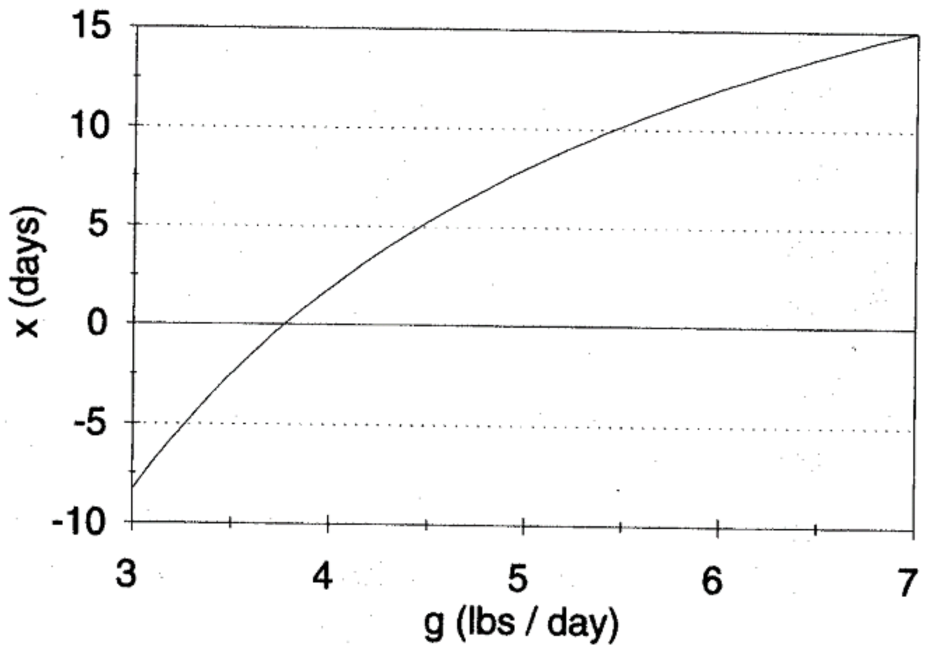
$$w = 200 + gt$$

$$f(x) = (0.65 - 0.01x)(200 + gx) - 0.45x$$

$$f'(x) = \frac{-(2gx + 5(49 - 13g))}{100} = 0$$

$$x = \frac{5(13g - 49)}{2g}$$

- As long as g>49/13, the optimal time is given by x



- We can define a *sensitivity function* as the relative changing of one value causing the relative changing of another value.

$$S(x,r) = lim\frac{\Delta x/x}{\Delta r/r} \rightarrow \frac{dx}{dr}\frac{r}{x}$$

- At the point r=0.01, x=8, we have

$$S(x,r) = \frac{-7}{25r^2}\frac{r}{x} = (-2800)(\frac{0.01}{8}) = \frac{-7}{2}$$

- Thus if r goes up by 2%, x will goes down by 7%.

- Similar, $S(x,g) = \frac{dx}{dg}\frac{g}{x} = 3.0625$

- Thus if 1% increase in the growth rate would cause to wait about 3% longer

- Question: what if the objective function is complicated or even not differentiable?
- Reconsider the pig selling problem, assume the growth rate of the pig is increasing with weight, when should we sell the pig for maximum profit?

$$Pr = w(t) * p(t) - c(t) = (200 + 5t)(0.65 - 0.01t) - 0.45t$$

- Remodel the weight growth of pig
- Assume the growth rate of the pig is proportional to its weight, that is, $\frac{dw}{dt} = cw$
- From the fact that dw/dt=5 lbs/day when w(0)=200 lbs, we have $w = 200e^{0.025t}$
- The objective function

$$y = f(x) = (0.65 - 0.01x)(200e^{0.025x}) - 0.45x$$

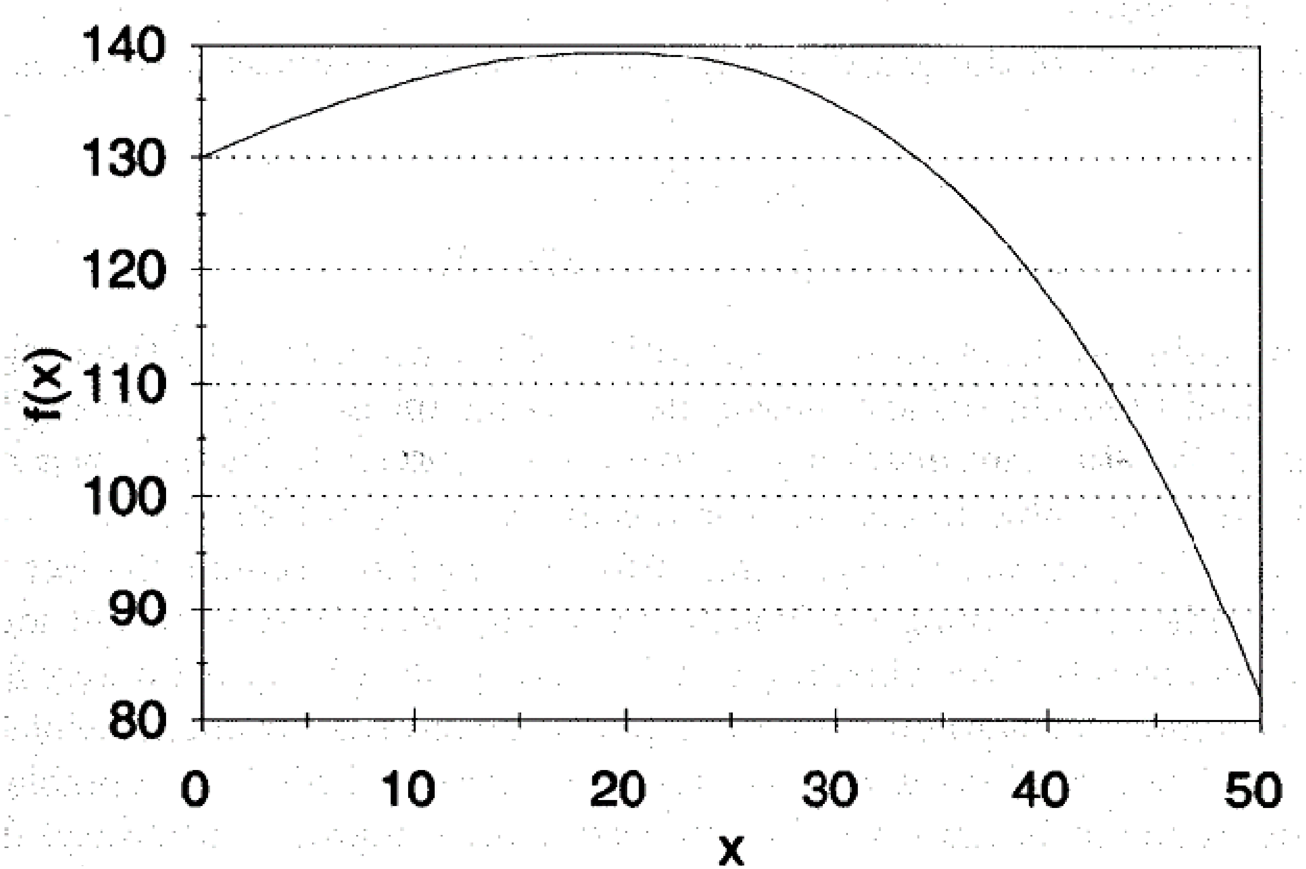
- If we use the previous method

$$f'(x) = 200 * 0.025e^{0.025x}(0.65 - 0.01x) - 2e^{0.025x} - 0.45 = 0$$

$$x = ?$$

Graphing Method

- Straight forward
- Maximum occurs at around x=20,y=140



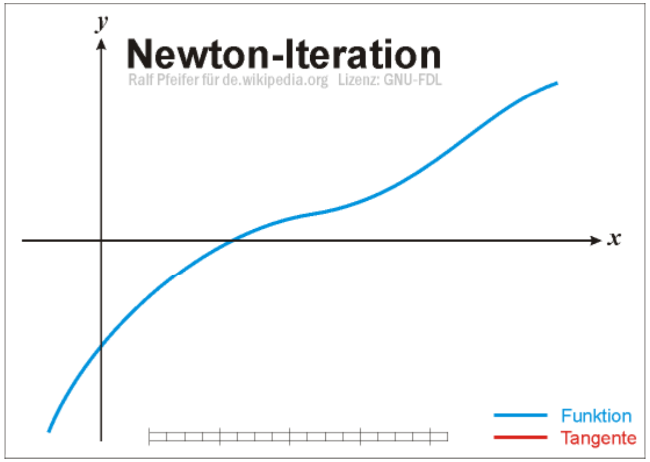
Newton's Method

- The idea of the method is as follows:
  - 1. Starts with an initial guess x_0
  - 2. Since $f'(x_n) = \frac{\Delta y}{\Delta x} = \frac{f(x_n) - 0}{x_n - x_{n+1}}.$,

    we can make a better estimation by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

  - 3. Repeat2 until f approaches to 0.



- In our case, let

$$dy/dx = 200 * 0.025e^{0.025x}(0.65 - 0.01x) - 2e^{0.025x} - 0.45 = 0$$

- Use Newton's method, we have

$$x \approx 19.5, y = 139.395$$

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Multivariable Optimization

Example: TV Manufacture Problem

- A factory manufactures color TVs of 19’ and 21’
- Cost: $195 per 19’ and $225 per 21’, plus a fixed cost $400,000
- Suggest price: $339 per 19’ and $399 per 21’
- Market impact:
  - Average price drops by 1 cent for each unit sold
  - Average price of 19’ reduces additional 0.3 cents for each 21’ sold
  - The average price of 21’ reduces additional 0.4 cents for each 19’ sold
- How many units of each type should be manufactured?

- Variables:

  s = number of 19' sold (*)

  t = number of 21' sold

  p = average selling price of 19'

  q = average selling price of 21'

  C = total cost of manufacture

  R = revenue from the sale of TVs

  P = profit

- Assumptions

  $C=400{,}000+195s+225t$

  $p=339-0.01s-0.03t$

  $q=399-0.01t-0.04s$

  $R=ps+qt$

  $P=R-C$

  $s\geq 0,\ t\geq 0$

- Objective: Maximize P

- Define:

$$\nabla f(x_1, \cdots, x_n) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$$

- Theorem: the extreme points satisfies $\nabla f = 0$
- The solution is obtained by solving the equations

$$\begin{aligned}
&\frac{\partial f}{\partial x_1}(x_1, \cdots, x_n) = 0 \\
&\vdots \\
&\frac{\partial f}{\partial x_n}(x_1, \cdots, x_n) = 0
\end{aligned}$$

- Reformulate the model
- Let y=P, x_1=s, x_2=t
- The problem is maximize

$$y = f(x_1, x_2) = (339 - 0.01x_1 - 0.03x_2)x_1$$
$$+(399 - 0.04x_1 - 0.01x_2)x_2 - (400000 + 195x_1 + 225x_2)$$

over the set S={x_1,x_2: $x_1 \geq 0$, $x_2 \geq 0$}

- Solve the equations

$$\frac{\partial f}{\partial x_1} = 144 - 0.02x_1 - 1 - 0.007x_2 = 0$$

$$\frac{\partial f}{\partial x_n} = 174 - 0.007x_1 - 0.02x_2 = 0$$

- We have

$$x_1 = 4735, x_2 = 7043, y = 553641$$

- Answer:
- The company can maximize profits by manufacturing 4735 of 19’ sets and 7043 of 21’ sets, resulting in a net profit for $553,641 for the year.

Constrained Optimization Problem

- Question: we have assumed unlimited number of TV sets, what if there is constraints?
- Assume
  - The number of 19' sets is limited by 5000 per year,
  - The 21' sets is limited by 8000 per year,
  - The total production capacity is 10000 per year
- How many units of each type should be manufactured?
- The previous optimal solution is no longer valid

$$x_1 = 4735, x_2 = 7043, y = 553641$$

Lagrange Multipliers

- Lagrange multipliers can be used to solve multivariable constrained optimization problem
- Standard form

$$
\begin{aligned}
&Max/min\ y = f(x_1, \cdots, x_n)\\
&s.t.\\
&g_1(x_1, \cdots, x_n) = c_1\\
&g_2(x_1, \cdots, x_n) = c_2\\
&\vdots\\
&g_k(x_1, \cdots, x_n) = c_k
\end{aligned}
$$

- Define: $\nabla f(x_1, \cdots, x_n) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$

- Theorem: if $\nabla g_1, \cdots, \nabla g_k$ are linear independent vectors, at an extreme point x, we must have

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k$$

- We call $\lambda_1, \cdots, \lambda_k$ the *Lagrange multipliers*, and $\nabla g_1, \cdots, \nabla g_k$ the gradient vectors.

- In order to locate the max-min points, we must solve the n Lagrange multiplier equations

$$\frac{\partial f}{\partial x_1} = \lambda_1 \frac{\partial g_1}{\partial x_1} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_1}$$
$$\vdots$$
$$\frac{\partial f}{\partial x_n} = \lambda_1 \frac{\partial g_1}{\partial x_n} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_n}$$

together with the k constraint equations

$$g_1(x_1, \cdots, x_n) = c_1$$
$$g_2(x_1, \cdots, x_n) = c_2$$
$$\vdots$$
$$g_k(x_1, \cdots, x_n) = c_k$$

Example

- Maximize $x+2y+3z$ over the set $x^2+y^2+z^2=3$.

$f(x,y,z)= x+2y+3z$

$g(x,y,z)= x^2+y^2+z^2$

Let $\nabla f = \lambda \nabla g$

We have

$1=2x\lambda$

$2=2y\lambda$

$3=2z\lambda$

together with $x^2+y^2+z^2=3$,

we obtain the max-min points.

- Back to the constrained TV Manufacture problem:
- Variables:

  s = number of 19’ sold (*)
  t = number of 21’ sold
  p = average selling price of 19’
  q = average selling price of 21’
  C = total cost of manufacture
  R = revenue from the sale of TVs
  P = profit

- Assumptions

  $C=400{,}000+195s+225t$
  $p=339-0.01s-0.03t$
  $q=399-0.01t-0.04s$
  $R=ps+qt$
  $P=R-C$
  $s \leq 5000$
  $t \leq 8000$
  $s+t \leq 10000$
  $s \geq 0,\ t \geq 0$

- Objective: Maximize P

- Choose the Lagrange multipliers method

- Reformulate the model
- Objective function

$$y = f(x_1, x_2) = (339 - 0.01x_1 - 0.03x_2)x_1$$
$$+(399 - 0.04x_1 - 0.01x_2)x_2 - (400000 + 195x_1 + 225x_2)$$

over the set S bounded by

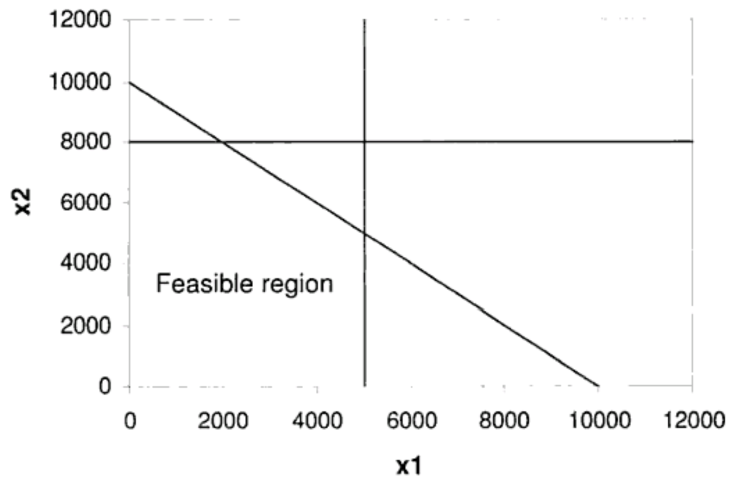
s≤5000, t≤8000, s+t≤10000, s≥0, t ≥0

- Compute

$$\nabla f = (144 - 0.02x_1 - 0.07x_2, 147 - 0.07x - 1 - 0.02x_2)$$

- Since $\nabla f \neq 0$ in S, the maximum must occur on the boundary

$$g(x_1, x_2) = x_1 + x_2 = 10000$$



- The Lagrange multiplier equations are

$$144 - 0.02x_1 - 0.007x_2 = \lambda$$
$$174 - 0.07x - 1 - 0.02x_2 = \lambda$$
$$x_1 + x_2 = 10000$$

- We have

$$x_1 = 3846, x_2 = 6154, y = 532308$$

- Try other boundaries, we have smaller value, thus y=532308 is the optimal solution

- Answer:
- The company can maximize profits by manufacturing 3846 of 19' sets and 6154 of 21' sets, resulting in a net profit for $532,308 for the year.

Homework

- 1. Give examples of one variable optimization and multivariable optimization according to research papers you read.
- 2. Try to use Matlab/Maple/Mathematica to solve the pig selling problem with Newton's method
- 3. (p50, ex-6)

6. A manufacturer of personal computers currently sells 10,000 units per month of a basic model. The cost of manufacture is \$700/unit, and the wholesale price is \$950. During the last quarter the manufacturer lowered the price \$100 in a few test markets, and the result was a 50% increase in sales. The company has been advertising its product nationwide at a cost of \$50,000 per month. The advertising agency claims that increasing the advertising budget by \$10,000/month would result in a sales increase of 200 units/month. Management has agreed to consider an increase in the advertising budget to no more than \$100,000/month.

   (a) Determine the price and the advertising budget that will maximize profit. Use the five–step method. Model as a constrained optimization problem, and solve using the method of Lagrange multipliers.